ELSEVIER

Discrete Mathematics 259 (2002) 237–250

DISCRETE MATHEMATICS

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Extremal problems on convex lattice polygons in sense of l_p-metrics

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Received 4 December 1999; received in revised form 3 December 2001; accepted 17 December 2001

Abstract

This paper expresses the minimal possible l_p-perimeter of a convex lattice polygon with respect to its number of vertices, where p is an arbitrary integer or $p=\infty$. It will be shown that such a number, denoted by $s_p(n)$, has $n^{3/2}$ as the order of magnitude for any choice of p. Moreover,

$$s_p(n) = \frac{2\pi}{\sqrt{54A_p}} n^{3/2} + \mathcal{O}(n),$$

where n is the number of vertices, A_p equals the area of planar shape $|x|^p + |y|^p \leqslant 1$, and p is an integer greater than 1. A consequence of the previous result is the solution of the inverse problem. It is shown that

$$N_p(s) = \frac{3\sqrt[3]{A_p}}{\sqrt[3]{2\pi^2}} s^{2/3} + \mathcal{O}(s^{1/3})$$

equals the maximal possible number of vertices of a convex lattice polygon whose l_p-perimeter is equal to s. The latter result in a particular case p=2 follows from a well known Jarnik's result. The method used cannot be applied directly to the cases $p=1$ and ∞. A slight modification is necessary. In the obtained results the leading terms are in accordance with the above formulas ($A_1 = 2$ and $A_\infty = 4$), while the rest terms in the expressions for $s_p(n)$ and $N_p(s)$ are replaced with $\mathcal{O}(n \log n)$ and $\mathcal{O}(s^{1/3} \log s)$, respectively. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Combinatorial optimization; Convex lattice polygon

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PII: S0012-365X(02)00384-9

1. Introduction

In this paper, we consider a class of convex lattice polygons with the property that they have the minimal l_p-perimeter with respect to the number of their vertices. In other words, if

$$s_p(n) = \min \left\{ \sum_{e \text{ is a edge of } Q} l_p\text{-}length\ of\ e\colon\ Q \text{ is a convex lattice } n\text{-gon} \right\},$$

then a convex lattice n-gon Q is said to be optimal if its l_p-perimeter is equal to $s_p(n)$. Such a polygon will be denoted by $Q_p(n)$. The main purpose of this paper is to describe the asymptotic behavior for $s_p(n)$, when $n \to \infty$ and p is a fixed integer greater than 1.

The inverse problem is to determine the maximal possible number of vertices of a convex lattice polygon if its l_p-perimeter is given. The result follows from the solution of the initial problem.

Let us mention here that a classical result, given by Jarnik in 1926 [7], implies the solution of the inverse problem in the case $p = 2$.

By a very slight modification of the method applied to $p = 2, 3, \ldots$ one can obtain the solutions (of both initial and inverse problems) in the case of l_1 and l_∞ metrics. The case $p = 1$ is equivalent to the problem studied in [1].

Let us mention here that extremal problems on the integer lattice (sometimes so called integer grid) play an important role in the area of image processing and pattern recognition. Namely, the integer grid is a mathematical model for binary pictures, while the convex shapes are the most studied shapes in these areas.

The rest of the paper is organized as follows. The basic definitions and denotations are given in Section 2. In Section 3, a solution of the problem is given for some special values of n, which form a monotonically increasing (unbounded) sequence. The problem is solved for the general case in Section 4. The inverse problem is studied in Section 5, while the comments and conclusion, including analysis of $p = 1$ and ∞ cases, are in the last section.

2. Preliminaries

A convex lattice polygon is a polygon whose vertices are points on the integer lattice and whose interior angles are strictly less than π radians (no three vertices are collinear). A polygon with n vertices will be called *n-gon*.

If a and b are integers, then $a \perp b$ means that a and b are relatively prime, while $\varphi(n)$ is the Euler function [2] which denotes the number of integers from $\{1, 2, \ldots, n\}$ which are relatively prime to n, particularly $\varphi(1) = 1$.

$U_p(n)$ is the partition function which counts the number of the solutions of the equation $n = x^p + y^p$, where x and y are positive, relatively prime integers and the order of numbers is taken into account. For example, $U_2(170)=4$, because $1^2+13^2=13^2+1^2=7^2+11^2=11^2+7^2=170$. Especially, $U_p(1)$ is defined to be equal to 1 (for every integer p).

$\mu(n)$ is the well-known Moebius function [2] defined as

(i) $\mu(1)=1$;
(ii) if $n>1$, let $n=p_1^{a_1}\cdots p_k^{a_k}$ be the prime decomposition of n. Then

$$\mu(n)=\begin{cases}(-1)^k & \text{if } a_1=a_2=\cdots=a_k=1,\\ 0 & \text{otherwise.}\end{cases}$$

Let $e=[(x_1,y_1),(x_2,y_2)]$ be an edge of the convex lattice polygon Q. Let us denote the differences $|x_2-x_1|$ and $|y_2-y_1|$ by $x(e)$ and $y(e)$, respectively. For practical reasons, we define the slope of e as $y(e)/x(e)$.

For a given integer t we define the set, $S_p(t)$, of the slopes in the following way:

$$S_p(t)=\left\{\frac{k}{l}:\ k^p+l^p\leqslant t,\ k\perp l,\ k \text{ and } l \text{ are integers}\right\}.$$

In the l_p-metrics ($p\geqslant 1$) the length of the edge e, usually denoted by $l_p(e)$, is defined to be $\sqrt[p]{x(e)^p+y(e)^p}$.

The l_p-perimeter of a polygon Q is

$$\mathrm{per}_p(Q)=\sum_{e \text{ is edge of } Q} l_p(e).$$

A_p will denote the area of the planar shape $|x|^p+|y|^p\leqslant 1$ for $p=1,2,\ldots,$ while $A_\infty=4$. $B_p(t)$ denotes the number of lattice points (different from the origin) inside $|x|^p+|y|^p\leqslant t$.

It is useful to introduce a sequence of integers in the following way:

$$n_p(t)=4\sum_{i=1}^{t}U_p(i),\quad t=1,2,3,\ldots\ .$$

Let us note that $n_p(t)=4\sum_{i=1}^{t}U_p(i)$ is a monotonically (not strictly) increasing, unbounded, function, with respect to t. So, for any integer n and a fixed integer p, the integer t satisfying $n_p(t-1)\leqslant n<n_p(t)$ exists uniquely.

3. Optimality in case $n=n(t)$

A lower bound for the l_p-perimeter of a convex lattice n-gon Q, with $n_p(t-1)\leqslant n<n_p(t)$, is established by the following lemma.

Lemma 1. *For integers n and t, satisfying* $n_p(t-1) \leqslant n < n_p(t)$, *the following inequality holds*:

$$s_p(n) \geqslant (n - n_p(t-1))\sqrt[p]{t} + 4\sum_{i=1}^{t-1} \sqrt[p]{i}U_p(i).$$

Proof. For a given integer n, let t be the integer determined by $n_p(t-1) \leqslant n < n_p(t)$ and let Q be an arbitrary convex lattice n-gon. There are no three parallel edges of Q, since Q is a convex polygon. So, there are at most four edges with the same slope (where the slope of the edge e is taken in the previously defined sense). Therefore, for any integer i, there are atmost $4U_p(i)$ edges with the l_p-length equal to $\sqrt[p]{i}$, if it is assumed that for any chosen edge e, $x(e) \perp y(e)$ is satisfied. A lower bound for the sum of the l_p-lengths of edges of Q can be obtained by taking:

- $4U_p(1) = 4 \cdot 1$ edges with the l_p-length 1,
- $4U_p(2) = 4 \cdot 1$ edges with the l_p-length $\sqrt[p]{2}$,
- $4U_p(3) = 4 \cdot 0$ edges with the l_p-length $\sqrt[p]{3}$,

...
- $4U_p(t-1)$ edges with the l_p-length $\sqrt[p]{t-1}$,
  and finally,
- $n - n_p(t-1) = n - 4\sum_{i=1}^{t-1} U_p(i)$ edges with the l_p-length $\sqrt[p]{t}$.

That completes the proof. □

Since the previous lower bound is established in a "greedy manner", it will be called the greedy lower bound and will be denoted by $glb_p(n)$:

$$glb_p(n) = (n - n_p(t-1))\sqrt[p]{t} + 4\sum_{i=1}^{t-1} \sqrt[p]{i}U_p(i), \tag{1}$$

where the integer t is determined uniquely by $n_p(t-1) \leqslant n < n_p(t)$.

Thus:

$$glb_p(n) \leqslant s_p(n), \quad \text{for } n \geqslant 3 \text{ and } p = 1, 2, \ldots .$$

Further, we now prove that in the cases $n = n_p(t)$ (for any integer t), the optimal polygon $Q_p(n_p(t))$ is uniquely determined. Let us note that, in general, there can be many convex lattice n-gons with the l_p-perimeter equal to $s_p(n)$.

Lemma 2. *For any integer t and a fixed integer p* $(p = 1, 2, \ldots)$ *there exists an optimal convex lattice* $n_p(t)$*-gon*, $Q_p(n(t))$, *whose* l_p*-perimeter is equal to* $4\sum_{i=1}^{t} \sqrt{i}U_p(i)$. *In other words*

$$s_p(n(t)) = \mathrm{per}_p(Q_p(n_p(t))) = 4\sum_{i=1}^{t} \sqrt[p]{i}U_p(i) = glb_p(n_p(t))$$

is satisfied.

The polygon $Q_p(n(t))$ is determined uniquely.

Proof. The proof is constructive. The polygon $Q_p(n_p(t))$ consists of four isometric arcs, whose edge slopes coincide with the set

$$S_p(t) = \left\{ \frac{k}{l}:\ k^p + l^p \leqslant t, \quad k \perp l, \quad k \text{ and } l \text{ are integers} \right\}.$$

More precisely, let a convex lattice $Q_p(n_p(t))$-gon be given, and let the lattice points $A_0 = (x_0, y_0), A_1 = (x_1, y_1), \dots, A_n = (x_{n_p(t)}, y_{n_p(t)}) = A_0$ be the counterclockwise ordered vertices of $Q_p(n_p(t))$.

Let $e_1, e_2, \dots e_{n_p(t)}$ be the edges determined by consecutive points from the previous sequence, i.e., $e_1 = A_0A_1, e_2 = A_1A_2, \dots, e_{n_p(t)} = A_{n_p(t)-1}A_{n_p(t)}$. Then, the edges $e_1, e_2, \dots, e_{n_p(t)}$ can be arranged into four arcs. If the angle between the positively oriented x-axis and the edge $A_{i-1}A_i$ is observed, then

- the south-east arc contains only the edges whose angles belong to $[0, \frac{\pi}{2})$;
- the north-east arc contains only the edges whose angles belong to $[\frac{\pi}{2}, \pi)$;
- the north-west arc contains only the edges whose angles belong to $[\pi, \frac{3\pi}{2})$;
- the south-west arc contains only the edges whose angles belong to $[\frac{3\pi}{2}, 2\pi)$.

The vertex A_0 is chosen to be one of the vertices, having the minimal y-coordinate, which has the minimal x-coordinate (the "left lowest" point) then the vertex $A_{(1/4)n_p(t)}$ is one of the vertices having the maximal x-coordinate, which has the minimal y-coordinate (the "lowest outermost right" point). For convenience and without loss of generality, let us assume $A_0 = (0,0)$. Since the slope of the edge e_i is equal to $y(e_i)/x(e_i)$ it follows that the vertices of the south-east arc of the polygon $Q_p(n_p(t))$ are:

$A_0 = (0,0)$,
$A_1 = (x(e_1), y(e_1)) = (0,1)$,
$A_2 = (x(e_1) + x(e_2), y(e_1) + y(e_2))$,
$\dots\dots$
$A_{(1/4)n_p(t))} = (x(e_1) + x(e_2) + \dots + x(e_{\frac{1}{4}n_p(t)}), y(e_1) + y(e_2) + \dots + y(e_{\frac{1}{4}n_p(t)}))$.

The slopes in the arc have to be arranged in the increasing order

$$\frac{0}{1} = \frac{y(e_1)}{x(e_1)} < \frac{y(e_2)}{x(e_2)} < \dots < \frac{y(e_{\frac{1}{4}n_p(t)})}{x(e_{\frac{1}{4}n_p(t)})}$$

and

$$S_p(t) = \left\{ \frac{0}{1} = \frac{y(e_1)}{x(e_1)}, \frac{y(e_2)}{x(e_2)}, \dots, \frac{y(e_{\frac{1}{4}n_p(t)})}{x(e_{\frac{1}{4}n_p(t)})} \right\}.$$

The other three arcs are obtained by rotating the south-east arc for $\pi/2, \pi$ and $3\pi/2$ radians around the point $(0, y(e_1) + y(e_2) + \cdots + y(e_{\frac{1}{4}n_p(t)}))$. The described convex lattice polygon is optimal, since its l_p-perimeter reaches the established lower bound $glb_p(n_p(t))$ (see (1)). The uniqueness follows from the fact that any other convex lattice polygon (which is not isometric with the previously described polygon) must have an edge with the edge slope which does not belong to the set $S_p(t)$. That leads to the increment of its l_p-perimeter for an amount of $\sqrt[p]{t+1} - \sqrt[p]{t}$, at least. So, it cannot be the optimal one. □

We shall use the following already known result from number theory ([8, Theorem 3.15]) which estimates the number of lattice points inside domains bounded by so-called Lamé's curve: $|x|^\beta + |y|^\beta = u$, where β is an arbitrary real number with $\beta \geqslant 2$.

Theorem 3. *The number of lattice points belonging to the area $|x|^\beta + |y|^\beta = u$, with a fixed $\beta \geqslant 2$ is*

$$C_\beta u^{2/\beta} + \mathcal{O}(u^{\omega_\beta}),$$

where

$$C_\beta = \int\int_{x^\beta + y^\beta \leqslant u} \mathrm{d}x\,\mathrm{d}y$$

while

$$\omega_\beta = \begin{cases} \frac{2}{3\beta} & \text{for } 2 \leqslant \beta \leqslant 3 \\ \frac{1}{\beta} - \frac{1}{\beta^2} & \text{for } \beta \geqslant 3. \end{cases}$$

Now, we can derive the asymptotic behavior for the functions $n_p(t)$ and $s_p(t)$, where $p \geqslant 2$ is an arbitrary integer.

Lemma 4. *For a given integer $p \geqslant 2$, the function $n_p(t)$ can be estimated by*

$$n_p(t) = 4 \sum_{q \leqslant t} U_p(q) = \frac{6A_p}{\pi^2} t^{2/p} + \mathcal{O}(t^{1/p}).$$

Proof. Let us note that $n_p(t)$ is the number of lattice points (x, y) satisfying $|x|^p + |y|^p \leqslant t$, where $x \perp y$. Since $B_p(t)$ denotes the number of lattice points (different from the origin) inside of $|x|^p + |y|^p \leqslant t$, we have

$$B_p(t) = n_p(t) + n_p\left(\frac{t}{2^p}\right) + n_p\left(\frac{t}{3^p}\right) + \cdots = \sum_{i=1}^{\lfloor \sqrt[p]{t} \rfloor} n_p\left(\frac{t}{i^p}\right).$$

That gives

$$B_p\left(\frac{t}{n^p}\right) = \sum_{m=1}^{\infty} n_p\left(\frac{t}{n^p m^p}\right).$$

By standard techniques we have:

$$\begin{aligned}
n_p(t) &= \sum_{l=1}^{\lfloor \sqrt[p]{t} \rfloor} n_p\left(\frac{t}{l^p}\right)\left(\sum_{n|l}\mu(n)\right) = \sum_{n=1}^{\lfloor \sqrt[p]{t} \rfloor} \mu(n)\left(\sum_{m=1}^{\lfloor \sqrt[p]{t} \rfloor} n_p\left(\frac{t}{n^p m^p}\right)\right)\\
&= \sum_{n=1}^{\lfloor \sqrt[p]{t} \rfloor} \mu(n) B_p\left(\frac{t}{n^p}\right) = \sum_{n=1}^{\lfloor \sqrt[p]{t} \rfloor} \mu(n)\left(A_p \frac{t^{2/p}}{n^2} + \mathcal{O}\left(\left(\frac{t}{n^p}\right)^{\omega_p}\right)\right)\\
&= A_p t^{2/p} \sum_{n=1}^{\lfloor \sqrt[p]{t} \rfloor} \frac{\mu(n)}{n^2} + \mathcal{O}\left(t^{\omega_p} \sum_{n=1}^{\sqrt[p]{t}} \frac{\mu(n)}{n^{p\omega_p}}\right)\\
&= A_p t^{2/p}\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} - \sum_{n=\lfloor \sqrt[p]{t} \rfloor}^{\infty} \frac{\mu(n)}{n^2}\right) + \mathcal{O}\left(t^{\omega_p} \int_1^{\sqrt[p]{t}} \frac{\mathrm{d}x}{x^{p\omega_p}}\right)\\
&= \frac{6A_p}{\pi^2} t^{2/p} + \mathcal{O}\left(t^{1/p}\right).
\end{aligned}$$

Let us note that Theorem 3,

$$\left|\sum_{n=\lfloor \sqrt[p]{t} \rfloor}^{\infty} \frac{\mu(n)}{n^2}\right| < \sum_{\lfloor \sqrt[p]{t} \rfloor}^{\infty} \frac{1}{n^2} = \mathcal{O}(t^{-1/p}),$$

and

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{\pi^2}{6}$$

are used in the last derivation. $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function, if s is a complex number with $Re(s) > 1$ (see [5]). □

Lemma 5. *The l_p-perimeter ($p = 2, 3, \ldots$) of optimal convex lattice polygons $Q_p(n_p(t))$ can be expressed as follows*

$$s_p(n_p(t)) = 4 \sum_{q \leqslant t} \sqrt[p]{q}\, U_p(q) = \frac{4A_p}{\pi^2} t^{3/p} + \mathcal{O}(t^{2/p}).$$

Proof. By the Stieltjes integration

$$\sum_{q\leqslant t} \sqrt[p]{q}U_p(q) = \int_1^t \sqrt[p]{u}\,\mathrm{d}(n_p(u)) = \sqrt[p]{u}n_p(u)|_1^t - \frac{1}{p}\int_1^t \frac{n_p(u)}{u^{1-1/p}}\,\mathrm{d}u$$

$$= \frac{6A_p}{\pi^2}t^{3/p} + \mathcal{O}(t^{2/p}) - \frac{6A_p}{p\pi^2}\int_1^t (u^{3/p-1} + \mathcal{O}(u^{2/p-1}))\,\mathrm{d}u$$

$$= \frac{4A_p}{\pi^2}t^{3/p} + \mathcal{O}(t^{2/p}). \qquad \square$$

The next lemma solves the initial problem of the paper in the case $n = n_p(t)$.

Lemma 6. *The asymptotic behavior of the perimeter of optimal convex lattice polygons $Q_p(n_p(t))$ expressed in function of the number of their vertices $n_p(t)$ is*

$$s_p(n_p(t)) = \frac{2\pi}{\sqrt{54A_p}}(n_p(t))^{3/2} + \mathcal{O}(n_p(t)), \quad \textit{for} \ p = 2, 3, \ldots \ .$$

Proof. By Lemma 4

$$n_p(t) = \frac{6A_p}{\pi^2}t^{2/p} + \mathcal{O}(t^{1/p})$$

which gives $t = \mathcal{O}((n_p(t))^{p/2})$. So,

$$\frac{\pi^2}{6A_p}n_p(t) + \mathcal{O}((n_p(t))^{1/2}) = t^{2/p}$$

implies

$$t = \left(\frac{\pi^2}{6A_p}n_p(t) + \mathcal{O}((n_p(t))^{1/2})\right)^{p/2}$$

$$= \frac{\pi^p}{(6A_p)^{p/2}}(n_p(t))^{p/2}\left(1 + \mathcal{O}\left(\frac{1}{(n_p(t))^{1/2}}\right)\right)^{p/2}$$

$$= \frac{\pi^p}{\sqrt{(6A_p)^p}}(n_p(t))^{p/2} + \mathcal{O}((n_p(t))^{(p-1)/2}).$$

($(1 + \mathcal{O}(\frac{1}{(n_p(t))^{1/2}}))^{p/2} = 1 + \mathcal{O}(\frac{1}{(n_p(t))^{1/2}})$ is used.)

The elimination of the parameter t from $s_p(n_p(t))$ (see Lemma 5) finishes the proof, i.e.,

$$
\begin{aligned}
s_p(n_p(t)) &= \frac{4A_p}{\pi^2}\left(\frac{\pi^p}{\sqrt{(6A_p)^p}}(n_p(t))^{p/2} + \mathcal{O}((n_p(t))^{(p-1)/2})\right)^{3/p} + \mathcal{O}(n_p(t)) \\
&= \frac{4A_p}{\pi^2}\left(\frac{\pi^p}{\sqrt{(6A_p)^p}}(n_p(t))^{p/2}\right)^{3/p}(1+\mathcal{O}((n_p(t))^{(p-1)/2-p/2}))^{3/p} \\
&\quad +\mathcal{O}(n_p(t)) \\
&= \frac{2\pi}{\sqrt{54A_p}}(n_p(t))^{3/2} + \mathcal{O}(n_p(t)).
\end{aligned}
$$

($(1+\mathcal{O}((n_p(t))^{(p-1)/2-p/2}))^{3/p} = 1+\mathcal{O}(n_p(t)^{-1/2})$ is used.) □

4. General case

Now, we can derive the asymptotic expression for $s_p(n)$ (i.e., in this case n is not assumed to be of the form $n = n_p(t)$).

Theorem 7. *The following asymptotic expression holds for any integer* $p \geqslant 2$:

$$s_p(n) = \frac{2\pi}{\sqrt{54A_p}}n^{3/2} + \mathcal{O}(n).$$

Proof. Let p be a fixed integer larger than 1. If an integer n is given, let us determine the integer t such that:

$$n_p(t-1) \leqslant n < n_p(t).$$

Lemma 4 gives the same asymptotic estimate $(6A_p/\pi^2)t^{2/p} + \mathcal{O}(t^{1/p})$ for both $n_p(t-1)$ and $n_p(t)$. Consequently, n can be estimated, also by

$$\frac{6A_p}{\pi^2}t^{2/p} + \mathcal{O}(t^{1/p}).$$

This implies

$$t = \frac{\pi^p}{\sqrt{(6A_p)^p}}n^{p/2} + \mathcal{O}(n^{p/2-1/2}).$$

Also, from the definition of $s_p(n)$, it follows easily:

$$s_p(n_p(t-1)) \leqslant s_p(n) \leqslant s_p(n_p(t)).$$

Since a direct application of Lemma 5 gives $(4A_p/\pi^2)t^{3/p} + \mathcal{O}(t^{2/p})$ as a common estimate for $s_p(n_p(t-1))$ and $s_p(n_p(t))$, we have

$$s_p(n) = \frac{4A_p}{\pi^2} t^{3/p} + \mathcal{O}(t^{2/p}).$$

The statement follows by replacing the parameter t by

$$t = \frac{\pi^p}{\sqrt{(6A_p)^p}} n^{p/2} + \mathcal{O}(n^{p/2-1/2})$$

in the last equality. □

5. Inverse problem

The inverse problem is considered in this section. The question is

What is the maximal possible number of vertices of a convex lattice polygon with respect to its l_p-perimeter ?

More precisely, if $N_p(s)$ is defined to be

$$N_p(s) = \max\{n \mid \text{there exists a convex lattice } n\text{-gon } Q \text{ with } per_p(Q) \leqslant s\},$$

we are looking for the behavior of $N_p(s)$, as $s \to \infty$, for any fixed integer p.

From the definition of functions $s_p(n)$ and $N_p(s)$ one can conclude that

$$s_p(N_p(s)) \leqslant s$$

is valid for any integer p.

Note 1. *The functions $s_p(n)$ and $N_p(s)$ are not mutually inverse*, *since the previous inequality is strict for some values of s.*

What we need here is the equality $N_p(s_p(n)) = n$.

Lemma 8. *The functions $N_p(n)$ and $s_p(n)$ satisfy the following functional equality*:

$$N_p(s_p(n)) = n$$

for any integer p.

Proof. From definition of $s_p(n)$ it follows that there exists a convex lattice n-gon with l_p-perimeter $s_p(n)$ which contradicts $N_p(s_p(n)) < n$.

Also, a convex lattice n-gon with l_p-perimeter (strictly) less than $s_p(n)$ does not exist (from the definition of $s_p(n)$), which excludes the possibility $N_p(s_p(n)) > n$. Namely, if there exists a convex lattice $(n+k)$-gon $(k > 0)$ with l_p-perimeter equal to $s_p(n)$, then another convex lattice n-gon with l_p-perimeter less than $s_p(n)$ can be constructed easily—it is enough to exclude arbitrary k-vertices. □

The next theorem describes the asymptotic behavior of $N_p(s)$, for $p \geqslant 2$.

Theorem 9. *Let an integer $p \geqslant 2$ be given. Then*

$$N_p(s) = \frac{3\sqrt[3]{A_p}}{\sqrt[3]{2\pi^2}} s^{2/3} + \mathcal{O}(s^{1/3}).$$

Proof. If an integer s is given, let us determine the integer n such that:

$$s_p(n) \leqslant s < s_p(n+1).$$

That is possible since $s_p(n)$ is an unbounded, monotonically increasing function. Further, we have

$$N_p(s_p(n)) \leqslant N_p(s) < N_p(s_p(n+1)) \Leftrightarrow n \leqslant N_p(s) < n+1$$
$$\Leftrightarrow N_p(s) = n.$$

Because $s_p(n) \leqslant s < s_p(n+1)$ and Theorem 7, we have

$$s = \frac{2\pi}{\sqrt{54A_p}} n^{3/2} + \mathcal{O}(n).$$

Since $n = \mathcal{O}(s^{2/3})$, it follows:

$$n = \left(\frac{\sqrt{54A_p}}{2\pi} s + \mathcal{O}(s^{2/3}) \right)^{2/3} = \frac{\sqrt[3]{54A_p}}{\sqrt[3]{4\pi^2}} s^{2/3} \left(1 + \mathcal{O}\left(\frac{1}{s^{1/3}} \right) \right)^{2/3}$$
$$= \frac{3\sqrt[3]{A_p}}{\sqrt[3]{2\pi^2}} s^{2/3} + \mathcal{O}(s^{1/3}).$$

That completes the proof. □

A special case (for $p = 2$) of the above theorem follows from well-known Jarnik's results (see [7]),

$$N_2(s) = \frac{3}{\sqrt[3]{2\pi}} s^{2/3} + \mathcal{O}(s^{1/3}),$$

since $A_2 = \pi$.

6. Comments and concluding remarks

It is a classical result [7], that if G is a strictly convex curve of a length s, then (under a smooth condition on G) the maximal number of integer points lying on G is equal to $(3/\sqrt[3]{2\pi})s^{2/3} + \mathcal{O}(s^{1/3})$. The exponent and the constant in the leading term are best possible. In [11] it is shown that the exponent $2/3$ can be decreased by imposing suitable smoothness condition on G. In particular, if G has a continuous third derivative with a sensible bound, the best possible value of the exponent lies between

Table 1
Numerical values for $s_p(n)$ and $N_p(s)$

p	$s_p(n)$	$N_p(s)$
1	$\approx 0.60459978n^{3/2}$	$\approx 1.39858223s^{2/3}$
2	$\approx 0.48240083n^{3/2}$	$\approx 1.62577821s^{2/3}$
3	$\approx 0.45487714n^{3/2}$	$\approx 1.69071571s^{2/3}$
4	$\approx 0.44402188n^{3/2}$	$\approx 1.71816056s^{2/3}$
5	$\approx 0.43858813n^{3/2}$	$\approx 1.73232251s^{2/3}$
10	$\approx 0.43059954n^{3/2}$	$\approx 1.75368243s^{2/3}$
100	$\approx 0.42755126n^{3/2}$	$\approx 1.76200796s^{2/3}$
1000	$\approx 0.42751661n^{3/2}$	$\approx 1.76210318s^{2/3}$
10000	$\approx 0.42751661n^{3/2}$	$\approx 1.76210319s^{2/3}$
∞	$\approx 0.4275166n^{3/2}$	$\approx 1.7621032s^{2/3}$

3/5 and 1/3 inclusive. The generalization of this result to higher dimensions is given in [10]. The reference [3] gives the following related result: If G is the graph of the function f, then the assumptions $f \in C^d([0,N])$, $|f| \leqslant N$, $|f'| \leqslant 1$, $f^d \neq 0$ in $[0,N]$ imply $|G \cap \mathbf{Z}^2| \leqslant c(\varepsilon_d)N^{1/2+\varepsilon_d}$, where $\varepsilon_d \to 0$ as $d \to \infty$. Especially, if $f \in C^\infty([0,1])$ is strictly convex, then $|tG \cap \mathbf{Z}^2| \leqslant c(f,\varepsilon)t^{1/2+\varepsilon}$ for every $\varepsilon > 0$ (tG is the dilation of G by factor t, $t \geqslant 1$). In view of the example $f(x) = \sqrt{x}$ the exponent 1/2 is best possible.

This paper does not consider the number of lattice points on strictly convex curves. It is focussed on extremal problems on convex lattice polygons. The derivation in [7] is also made by construction of convex lattice polygons optimal in the sense of the Euclidean metrics (l_2-metrics) i.e., the sequence of convex lattice polygons having the Euclidean perimeter equal to s and whose number of vertices equals $(3/\sqrt[3]{2\pi})s^{2/3}$ within an error term of $\mathcal{O}(s^{1/3})$. This paper shows that if the perimeter s of an optimal polygon is taken in the sense of l_p-metrics, where p is an integer bigger than 1, the exponent in the leading term still remain $\frac{2}{3}$ while the constant should be changed depending on p. Some numerical values are given in Table 1.

Let us mention here that the method used here cannot be directly applied to the case $p = 1$. Namely, in the proof of Lemma 4, the number of lattice points inside of $|x|^p + |y|^p \leqslant t/n^p$ (denoted by $B_p(t/n^p)$) is not studied by Theorem 3 if p equals 1. The best which can be used in that case is

$$B_1\left(\frac{t}{n}\right) = A_1 \frac{t^2}{n^2} + \mathcal{O}\left(\frac{t}{n}\right) = 2\frac{t^2}{n^2} + \mathcal{O}\left(\frac{t}{n}\right),$$

since $|x| + |y| = t/n^p$ is a square (for more details about the number of lattice points inside of a planar domain which is blown up by a large factor we refer to [4,9]). This estimate (if the rest of the method remains the same) leads to the next results.

Theorem 10. *Asymptotic estimates for $s_1(n)$ and $N_1(s)$ are*

$$s_1(n) = \frac{2\pi}{\sqrt{54A_1}} n^{3/2} + \mathcal{O}(n \log n) = \frac{\pi}{3\sqrt{3}} n^{3/2} + \mathcal{O}(n \log n),$$

and

$$N_1(s)=\frac{3\sqrt[3]{A_1}}{\sqrt[3]{2\pi^2}}s^{2/3}+\mathcal{O}(s^{1/3}\log s)=\frac{3}{\sqrt[3]{\pi^2}}s^{2/3}+\mathcal{O}(s^{1/3}\log s).$$

A similar situation is in case of $p=\infty$. Noticing that $A_\infty=4$, while $U_\infty(n)$ should be understood to be equal to $2\varphi(n)$, one can obtain:

Theorem 11. *The following expressions hold*:

$$s_\infty(n)=\frac{2\pi}{\sqrt{54A_\infty}}n^{3/2}+\mathcal{O}(n\log n)=\frac{\pi}{3\sqrt{6}}n^{3/2}+\mathcal{O}(n\log n),$$

and

$$N_\infty(s)=\frac{3\sqrt[3]{A_\infty}}{\sqrt[3]{2\pi^2}}s^{2/3}+\mathcal{O}(s^{1/3}\log s)=\frac{6}{\sqrt[3]{4\pi^2}}s^{2/3}+\mathcal{O}(s^{1/3}\log s).$$

We conclude this paper with a notice that solutions of related problems can be of practical importance, because the integer grid is a mathematical model for computer picture (binary picture) and because the convex shapes are of a special interest in the area of image processing and pattern recognition. Let us give two illustrations (for more details see [1,6]).

(a) The maximal number of vertices of a convex lattice polygon which can be inscribed into the integer lattice of the size $m\times m$ (equivalently to the problem studied here for $p=1$) is an input parameter in the "worst case" complexity analysis of many algorithms on binary pictures of the size $m\times m$.
(b) The number of bits required for coding of digital convex polygons from the integer grid of the size $m\times m$, (or equivalently, the maximal number of convex lattice polygons which can be inscribed integer grid of the same size) describes the storage complexity for the convex shapes from binary $m\times m$ pictures. The number of bits required for the coding is $\mathcal{O}(m^{2/3})$ per a coded polygon but the description of an algorithm for such coding is still an open problem.

Acknowledgements

The author is thankful to the referees for useful suggestions which led to improvements of the paper.

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